‘Cart-Inverted Pendelum Control and Feedback Control’

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Abstract - In this case study, the basics of control theory are implemented and several systems were developed and analyzed. An inverted pendelum on a controlled cart system was successfully modeled in MATLAB and the pendelum was succesffully balanced using pole placement techniques and linearization around the balancing point. Furthermore in the second part of the case study, an airplane roll dynamics was successfully modeled and controlled in simulink and visualized with graphics.

I. BACKGROUND

Control Systems was historically a subfield of Electrical Engineering, but as the field has matured and innovation took off, it became its own specialty with a lot of demand for control knowledge and skills. In modern times control theory is used from everything from rocket launches and satellite control to controlling the temperature in a house, and in recent years has become increasingly prevalent in automotive design and marketing. The goal of Control Systems Engineers is to design systems that enable the control of dynamical and unpredictive phenomena instead of just being able to observe it. Once a system is built to describe some dynamical system, control theory is used to transform it into a system that can easily be controlled and meets the specifications of the original mission statement and goals. The control systems field is only growing and increasingly so as more and more systems are integrated together in our increasingly connected and complex world.

II. METHODS & RESULTS

The goal of this case study is to use state space models linearization and pole placement to control several systems.

A. *Cart Inverted-Pendelum Describtion*

The Inverted-Pendulum System has some system parameters that will be used repeatedly in this case study. The parameters as well as a graphics of the system can be seen in Figure 1.

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m1=1; m2=0.3; l=0.5; g=9.81;
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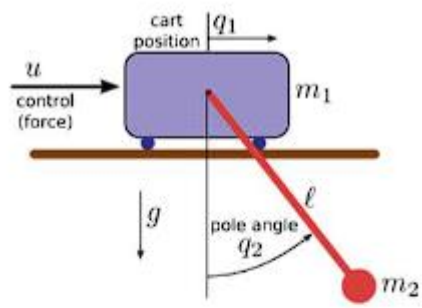

Figure 1 Shows a model of the cart inverted pendulum system.

It should be noted that the only control force that is given is the *u* which represents horizontal force on the cart. Even though the force *u* does not directly act on the pendulum, by moving the cart you indirectly move the pendulum. When these dynamic relationships are found, the pendulum can be controlled and balanced by only using the control force *u*.

B. *State Space Models*

State Space Models are models that use chosen states of the system to describe a system using a set of first-order differential equations instead of just one high order differential equation. This makes it easier to view a system as a collection of parts instead of one big complex unapproachable system. This also makes it easier to isolate, analyze, and change certain aspects of the system much easier.

The equations for the acceleration of the cart and pendulum are shown in Figure 2 and were used to construct the space state model in equation 1. These were derived from classical physics.

$$\ddot{q}_1 = \frac{\ell\, m_2 \sin(q_2)\, \dot{q}_2^2 + u + m_2\, g\, \cos(q_2)\, \sin(q_2)}{m_1 + m_2\left(1 - \cos^2(q_2)\right)}$$

$$\ddot{q}_2 = -\frac{\ell\, m_2 \cos(q_2)\, \sin(q_2)\, \dot{q}_2^2 + u\, \cos(q_2) + (m_1 + m_2)\, g\, \sin(q_2)}{\ell\, m_1 + \ell m_2\left(1 - \cos^2(q_2)\right)}$$

Figure 2 Shows the acceleration equations.

$$\mathrm{x} = \begin{pmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}, \dot{\mathrm{x}} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} \qquad (1)$$

After the state space model is set up in equation 1, the acceleration equations can be substituted in to give a more complete picture. Each of the state variables could have been replaced with x1, x2, x3, x4, but keeping them in their original scripts offers more clarity.

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dfrac{l\, m_2 \sin(q_2)\, \dot{q}_2^{\,2} + u + g\, m_2 \cos(q_2) \sin(q_2)}{m_1 - m_2\, \sigma_1} \\ -\dfrac{l\, m_2 \cos(q_2) \sin(q_2)\, \dot{q}_2^{\,2} + u \cos(q_2) + g \sin(q_2)\, (m_1 + m_2)}{l\, m_1 - l\, m_2\, \sigma_1} \end{pmatrix}$$

where

$$\sigma_1 = \cos(q_2)^2 - 1$$

Figure 3 Shows the state space model system.

C. Graphing the Cart Pendelum system

Using the code provided, a graphic can be obtained of the cart pendulum system, this will become even more important when the control is added to the system. Figure 4 shows the pendulum swinging back and forth after being solved using the ode45 solver in MATLAB. The ode45 function runs the state space model with respect to time and gives a vector of the values of the states at every sample time, this makes it very easy to graph as well as to debug. It's important to note that the input *u* is currently 0.

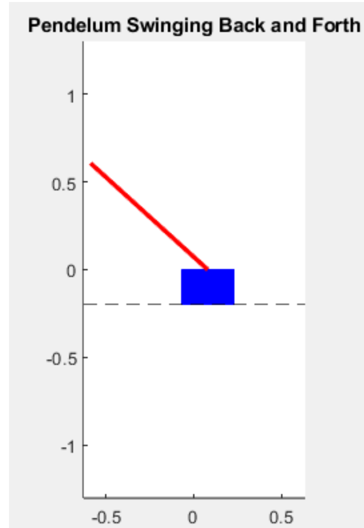

Figure 4 Shows a graphic of the pendulum swinging.

D. Adding Control Through From Input File

Having gained the ability to graph the system, it's time to add control to the system in order to control it. Before creating the system will decide the control, a file was given with the study that has control inputs at 100 Hz for 5 seconds, this input waveform can be seen in Figure 5.

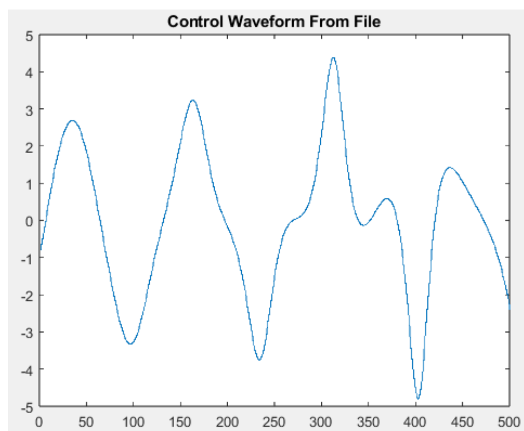

Figure 5 Shows control inputs for the system.

The graphic from Figure 5 was achieved using the stairs command in MATLAB and after running the ode45 solver with the control waveform, the pendulum became balanced, as can be seen from Figure 6. It's important to note that because the timing of the sample file may not line up with the timing of the ode45 solver, the two timings were interpolated at each point to get the most accurate results. This was done in a separate function and then passed into the ode45 solver.

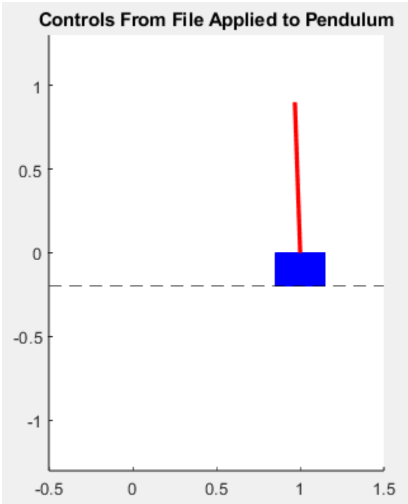

Figure 10 Shows the balanced pendulum resulting from control file.

$$\begin{pmatrix} 0 \\ 0 \\ -\dfrac{10}{3\cos(q_2)^2 - 13} \\ \dfrac{20\cos(q_2)}{3\cos(q_2)^2 - 13} \end{pmatrix}$$

Figure 7 The Jacobian with respect to u.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

Figure 6 B matrix when point is inputted.

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{2943}{1000} & 0 & 0 \\ 0 & \dfrac{12753}{500} & 0 & 0 \end{pmatrix}$$

Figure 8 A matrix for the linearized system.

$$\begin{pmatrix} 0 \\ 0 \\ -\dfrac{3\sqrt{7085}}{50} \\ \dfrac{3\sqrt{7085}}{50} \end{pmatrix}$$

Figure 9 Eigen values of the A matrix.

E. Linearization

Thanks to the control file given, the pendulum was successfully balanced, but it's possible to balance the system without any external files. By analyzing the system, the relationship between the control and the desired position can be derived.

To do this, the A matrix that describes the state space in terms of each state must be constructed. Because this is a nonlinear system thanks to the nonlinear terms in the acceleration equations, the system must be linearized first. This system was linearized by taking the Jacobian, a matrix constructed by taking the derivative of each state space equation by each of the state variables, this was done in MATLAB and the result can be seen in Figure 11.

To linearize the system, a point to linearize at must be chosen for the inverted pendulum system. There are equilibrium points at [0;0;0;0], when the pendulum is stationary at the bottom, and at [0;π;0;0] when the pendulum is completely balanced at the top of the cart. Since the goal is to balance at the top, the point [0;π;0;0] will be plugged into the Jacobian to finally reach the A Matrix for the linearized system. Figure 10 shows the A Matrix after the equilibrium point was plugged in. The Jacobian was also taken with respect to *u* in order to find the B Matrix shown in Figure 6 and Figure 7. As can be seen from Figure 9, one of the eigenvalues of the A Matrix is positive with signifies instability, this will have to be fixed before the pendulum can be balanced.

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\dfrac{150\,\dot{q}_2^{\,2}\cos(q_2) + 2943\cos(q_2)^2 - 2943\sin(q_2)^2}{\sigma_1} - \dfrac{600\cos(q_2)\sin(q_2)\,\left(150\sin(q_2)\,\dot{q}_2^{\,2} + 1000\,u + 2943\cos(q_2)\sin(q_2)\right)}{\sigma_1^{\,2}} & 0 & -\dfrac{300\,\dot{q}_2\sin(q_2)}{\sigma_1} \\ 0 & \dfrac{150\,\dot{q}_2^{\,2}\cos(q_2)^2 - 150\,\dot{q}_2^{\,2}\sin(q_2)^2 + 12753\cos(q_2) - 1000\,u\sin(q_2)}{\sigma_2} + \dfrac{300\cos(q_2)\sin(q_2)\,\left(150\cos(q_2)\sin(q_2)\,\dot{q}_2^{\,2} + 12753\sin(q_2) + 1000\,u\cos(q_2)\right)}{\sigma_2^{\,2}} & 0 & \dfrac{300\,\dot{q}_2\cos(q_2)\sin(q_2)}{\sigma_2} \end{pmatrix}$$

where

$$\sigma_1 = 300\cos(q_2)^2 - 1300$$

$$\sigma_2 = 150\cos(q_2)^2 - 650$$

Figure 11 Shows the Jacobian of the system, the A Matrix.

F. *Pole Placing and K Matrix*

Before continuing to making the system stable and controllable, the controllability of the matrix should be checked as it is good practice. Figure 12 shows the controllability matrix with rank 4 and since that is how many states are present, this system is indeed controllable.

$$\begin{matrix} 0 & 1.0000 & 0 & 5.8860 \\ 0 & 2.0000 & 0 & 51.0120 \\ 1.0000 & 0 & 5.8860 & 0 \\ 2.0000 & 0 & 51.0120 & 0 \end{matrix}$$

Figure 12 Shows the controllability matrix.

Now that the system is validated to be controllable, the poles/eigenvalues of the system must be changed to be made stable and converging. Using the place function in MATLAB, the K vector shown in Figure 13 was calculated based on the desired eigenvalues of [-1.1; -1.2; -1.3;-1.4].

```
k = 1×4
    -0.1224   17.4892   -0.3950    2.6975
```

Figure 13 The K matrix

The input u will now be transformed as seen in equation 2. Now the eigenvalues of the transformed linearized system are negative, and the system is ready to be tested graphically.

$$\boldsymbol{u = -k \cdot x\,,\ \dot{x} = (A - Bk)x} \quad \textbf{(2)}$$

G. *Putting It Together, Balanced Pendulum*

Running the ode45 solver with the static feedback controller allows the pendulum to remain balanced as it heads towards a desired point. When starting the pendulum slightly off the vertical position, the system successfully balanced the pendulum as can be seen in Figure 14. Not only can the pendulum be balanced, but it can also change the position of the cart across the horizontal axis while the pendulum is balanced as can also be seen in Figure 14, the desired cart position is set to one and it is successful in achieving that position. The system can shift form the equilibrium point on which the system was linearized partly because in the K Matrix the $\dot{q}_1 = -0.1224$ and the $\dot{q}_2 = 17.4892$, meaning that the system prioritizes the position of the pendulum much more than that of the cart. The gain on the position of the pendulum is 143 times greater than that of the position of the cart.

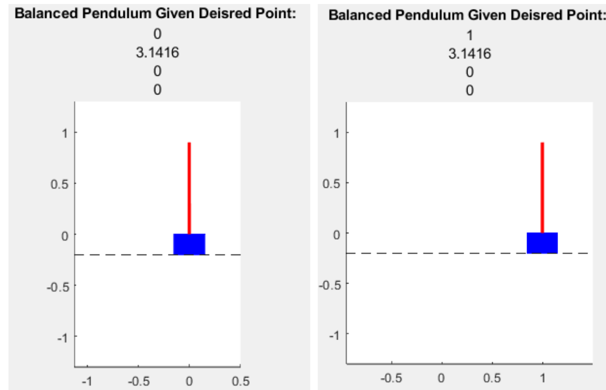

Figure 14 Shows the balanced pendulum given target point.

In order to balance the pendulum from any position, a function can be made to swing the pendulum up near enough to the equilibrium point to allow for the balancing controller to take over.

III. Project 2: Roll Dyanmics of an Airplane

A. *Background*

Similar to the inverted pendulum cart system, now the goal is to stabilize the roll of an airplane when subjected to a step input. A simple Simulink model was built to model this and can be seen in Figure 15.

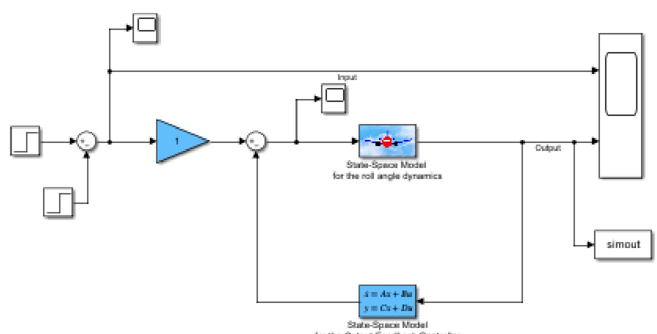

Figure 15 Shows the Simulink model for an aircraft roll.

As can be seen in Figure 15, the feedforward gain and the feedback system must be calculated and implemented.

B. *Feedback system*

The State Space model of the system was already provided, and its matrices can be seen in Figure 16. The eigenvalues of the A Matrix were calculated to be zeros and as can be seen in Figure 17, the step response of the system shows how unstable it is as the response grows indefinitely.

$$\dot{X} = \overset{A}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} X + \overset{B}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} u$$

$$Y = \overset{C}{\begin{bmatrix} 1 & 0 \end{bmatrix}} X + \overset{D}{0}$$

Figure 16 Shows the State Space Model of the Aircraft Roll.

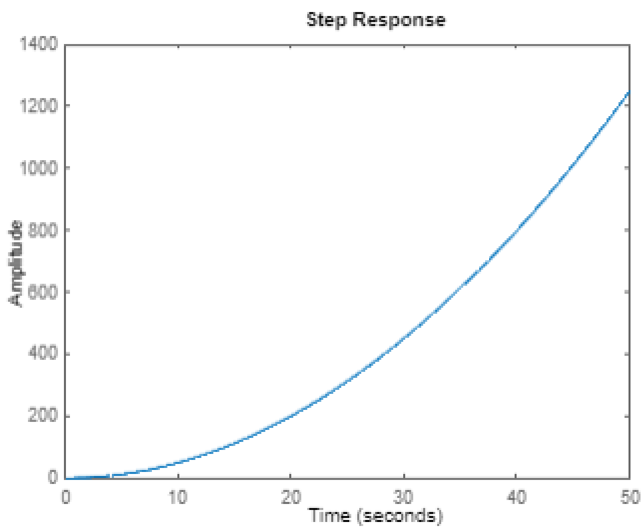

Figure 17 Shows the step response of the original system.

Now to make the system stable, the eigenvalues must be made negative and the new poles found. Because finding the ideal pole placement is difficult, the Simulink model was experiment with until the most ideal step response was found shown in Figure 18.

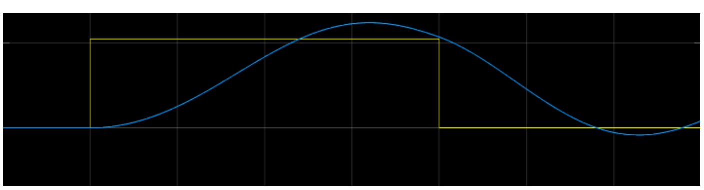

Figure 18 Shows the system step response.

The system parameters found experimentally can be seen in Figure 19. And after some manipulation to find the dc gain, the step response and the bode plot were graphed for the feedback system and can be seen in Figures 19 and 20 respectively.

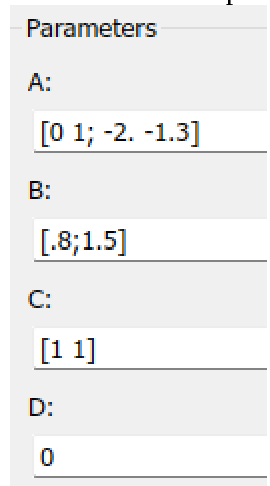

Figure 19 Shows System Parameters found from trial and error.

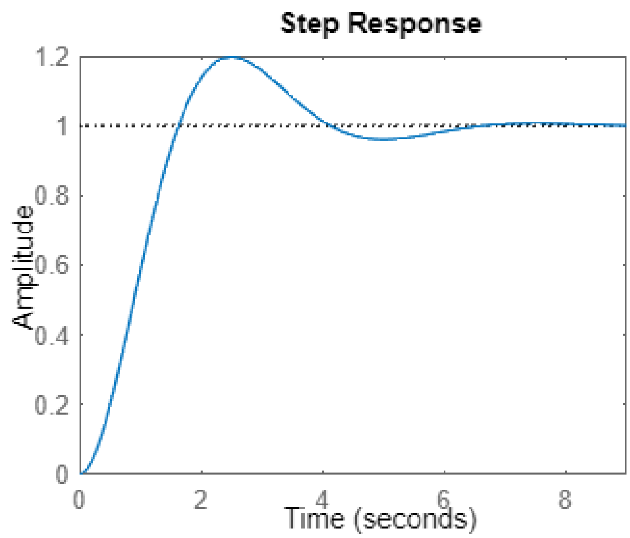

Figure 20 Step Response of stable system.

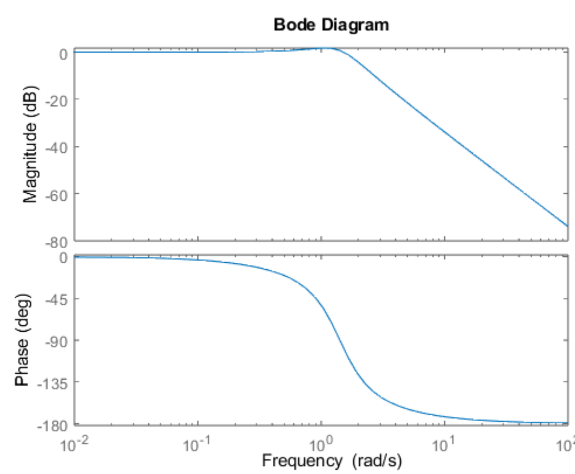

Figure 21 Bode Plot of Stable System.

As can be seen from the Bode Plot in Figure 21, the stable system behaves as a low pass filter only letting in very low frequencies. It can also be seen from the bode plot that the system is relatively stable since the magnitude is below 0 beginning from the 0 frequencies and is negative when the phase is above -180, a positive gain margin. The feedforward gain was set to 0.5 as that gave the best result.

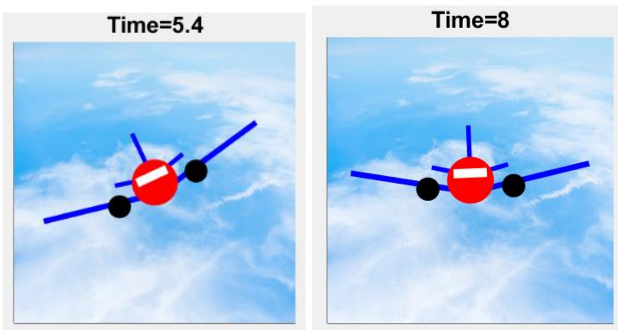

Figure 22 Shows the airplane turning and stabilizing.

IV. CONCLUSION

Control theory can be very powerful especially when used in tandem with powerful software like MATLAB. With several lines of code, an inverted pendulum on a moving cart can be balanced perfectly almost like magic. In this case study the concepts of State Space Models, Linearization, and pole placement were thoroughly developed through the use of several real-life physical systems. The pendulum was successfully balanced, and the plane turned relatively smoothly. More research is needed in the use of bode plots in analyzing stability and the big question of how to choose the new poles for the system.

V. REFRENCES

A. V. Oppenheim, A. S. Willsky, and H. S. Nawab, Signals and systems. London: Prentice-Hall, 1997.

“Matlab,” MATLAB Documentation. [Online]. Available:https://www.mathworks.com/help/matlab/. [Accessed: 22-Dec-2022].

Zeng, Shen, “ESE 441 “Final Exam” Project.

VI. APENDIX

Videos with live links:

Cart_Pendelum_Animation.mp4

Cart_Pendulum_Input_File_Animation.mp4

Balanced_Pendulum_To_Desired_Point.mp4

Plane_Roll_Animation.mp4