ESE 4481 Assignment #3 / Seif Elkhashab / 02-28-202

Dynamics Derivation of the Drone

Introduction:

This assignment seeks to walk through the derivation of conservation of translational and angular momentum for the UAV. The dynamics are derived i a frame centered at the c.g. of the UAV and rotating with the aircraft's principle axes, the body frame. The derivation in this homework assignment makes practical assumptions. A simplified dynamics for the quadcopter may satisfy the following 5 assumptions [1]:

- The body has a fixed mass distribution and constant mass;
- The air is at rest relative to the earth;
- The earth is fixed in inertial space;
- Flight in the earth atmosphere is close to the surface of earth, so the earth surface can be approximated as flat;
- Gravity is uniform so that the aircraft c.g. and center of mass are coincident, and gravitational forces do not change with altitude.

Notation to be Used:

-- X is a force in the x body axis. Y is a force in the y body axis. Z is a force in the z body axis. L is a moment about the body x axis. M is a moment about body y axis. N is a moment about the body z axis.

-- Assume the standard aerospace notation discussed in class holds with $(Z - Y - X)$ euler angles. ϕ is the roll euler angle between the earth NED (North, East, Down) frame and the body frame. θ is the pitch euler angle between the earth frame and the body frame. ψ is the yaw euler angle between the earth frame and the body frame.

-- Recall that a vector in the body frame p^b can represented as follows:

$$(1) \quad p^b = R_E^b p^E = R(\phi)R(\theta)R(\psi)p^E$$

where p^E is the vector as measured in the earth frame.

-- Assume that V^b is the airspeed vector as measured in the body frame with components as follows:

$$(2) \quad V^b = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

where u is the velocity component along the longitudinal axis, v is the velocity component along the lateral axis, and w is the velocity component along the vertical axis.

Translational and Angular Moment Balance in the Body Frame:

-- The linear momentum of the UAV is P,

$$(3) \quad P = mv,$$

and the angular momentum of the UAV is

$$(4) \quad H = J\omega.$$

-- For the time being, assume that the rotors do not spin and are also fixed so that they do not contribute to angular momentum. Newton's 2nd law applied to a quadcopter can be written in the earth frame (inertial frame) as follows:

$$(5) \quad \vec{F}^{E} = \left(\frac{d}{dt}(P)\right)^{E},$$

$$(6) \quad \vec{M}^{E} = \left(\frac{d}{dt}(H)\right)^{E}$$

where m is the mass of the UAV, V is the translational velocity of the body, ω_b/E is the angular velocity of the body, and $\vec{F}^{E}$ and $\vec{M}^{E}$ represent applied moments and forces,

$$(7) \quad \vec{F}^{E} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \vec{M}^{E} = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

-- One can rewrite (9) in a frame that rotates relative to the earth frame, the body frame. Assume that the centers of the body frame and the earth frame are coincident, but the body frame rotates relative to the earth frame by $\omega_{b/E}$. These laws now take the form under the assumption of constant mass and mass distribution:

$$(8) \quad \dot{V}^b = \frac{1}{m}\left(\vec{F}^{b} - \omega_{b/E} \times mV^b\right)$$

$$(9) \quad \dot{\omega}_{b/E} = J^{-1}\left(\vec{M}^{b} - \omega_{b/E} \times J\omega_{b/E}\right)$$

<u>Problem.</u> These equations are incomplete because they do not model the angular momentum of the propellers. Assume that each propeller has a mass moment of inertia of J_p and is offset from the center of mass. Two propellers spin CCW and two spin CW. The 4 propellers are organized on the 4 corners of a square with area l^2 and the center of mass of the drone is coincident with the center of the square. Each propeller spins at n_i Hz where i is the ith propeller

<u>(1) Derive an expression for the total angular moment of the drone.</u>

The total angular momentum will be the sum of angular momentum of the drone from (4) added with the angular momentum of each propeller. Since 2 propellers spin clockwise and 2 spin counter clockwise, the equation below accurately represents the total angular momentum of the drone.

$$H_{total} = J\omega + J_p(2\pi n_1 - 2\pi n_2 + 2\pi n_3 - 2\pi n_4)$$

<u>(2) Revise (9) so that it correctly accounts for angular momentum on the propellers.</u>

The original equation is

$$\dot{\omega}_{b/E} = J^{-1}\left(\overrightarrow{M}^{b} - \omega_{b/E} \times J\omega_{b/E}\right)$$

Adding the 4 propeller moments of inertia will involve more than just adding the 4 moments to the moment of the body of the drone since they are offset from he drone's center of mass. We need to take into account the gyroscopic torque from the propopllers. We already calculated the total angular momentume

$$H_{total} = J\omega + J_p(2\pi n_1 - 2\pi n_2 + 2\pi n_3 - 2\pi n_4)$$

substituting into the main equaiton

$$\dot{\omega}_{b/E} = J^{-1}(\overrightarrow{M}_b - \omega_{b/E} \times H_{total})$$

<u>(3) Code these equations in symbolic matlab. They will be used later in the assignment.</u>

The Inertial Matrix:

$$\mathrm{J} = \begin{pmatrix} J_x & 0 & -J_{\mathrm{xz}} \\ 0 & J_y & 0 \\ -J_{\mathrm{xz}} & 0 & J_z \end{pmatrix}$$

The Rotational Dynamics:

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} L - \frac{J_{\mathrm{xz}}\,\sigma_3}{\sigma_1} - \frac{J_z\,\sigma_2}{\sigma_1} \\ M - \frac{\omega_z\,(J_x\,\omega_x - J_{\mathrm{xz}}\,\omega_z) + \omega_x\,(J_{\mathrm{xz}}\,\omega_x - J_z\,\omega_z)}{J_y} \\ N - \frac{J_x\,\sigma_3}{\sigma_1} - \frac{J_{\mathrm{xz}}\,\sigma_2}{\sigma_1} \end{pmatrix}$$

where

$$\sigma_1 = {J_{\mathrm{xz}}}^2 - J_x\,J_z$$

$$\sigma_2 = \omega_y\,(J_{\mathrm{xz}}\,\omega_x - J_z\,\omega_z) + J_y\,\omega_y\,\omega_z$$

$$\sigma_3 = \omega_y\,(J_x\,\omega_x - J_{\mathrm{xz}}\,\omega_z) - J_y\,\omega_x\,\omega_y$$

Rotational Kinematics:

Recall that the rate of change of the euler angles $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$ is not the same as the body axis rotational rate that would be measured in a gyroscope p, q, and r. The rate of change of euler angles can be thought of as the magnitude of angular velocity vectors for the Earth axis [2].

$$(10) \quad \dot{\phi}\begin{bmatrix}1\\0\\0\end{bmatrix},\ \dot{\theta}\begin{bmatrix}0\\1\\0\end{bmatrix},\ \dot{\psi}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Changing the sum of the vectors rotated to the body axis frame, form the familiar roll rate, pitch rate, and yaw rate. This is done as follows (recalling the Z-Y-X order):

$$(11) \quad \begin{bmatrix}\dot{p}\\\dot{q}\\\dot{r}\end{bmatrix} = R(\phi)\begin{bmatrix}\dot{\phi}\\0\\0\end{bmatrix} + R(\phi)R(\theta)\begin{bmatrix}0\\\dot{\theta}\\0\end{bmatrix} + R(\phi)R(\theta)R(\psi)\begin{bmatrix}0\\0\\\dot{\psi}\end{bmatrix}$$

Let the vector Θ represent the euler angles,

$$(12) \quad \Theta = (\phi, \theta, \psi)^T$$

This yields the following expression that relates the rate of change of euler angles x to the angular velocities roll rate, pitch rate, and yaw rate $\omega_{b/E} = (p, q, r)^T$.

$$(13) \quad \omega_{b/E} = \begin{bmatrix}1 & 0 & -sin(\theta)\\0 & cos(\phi) & sin(\phi)cos(\theta)\\0 & -sin(\phi) & cos(\phi)cos(\theta)\end{bmatrix}\begin{bmatrix}\dot{\phi}\\\dot{\theta}\\\dot{\psi}\end{bmatrix} = H(\Theta)\dot{\Theta}$$

where H is the matrix in (6.2). To calculate the rate of change of euler angles in terms of angular rates the equation is inverted as follows

$$(14) \quad \dot{\Theta} = H(\Theta)^{-1}\omega_{b/E}$$

<u>Problem.</u> These are the rotational kinematical equations and we must integrate and control them to keep track of a drone's attitude.

<u>(1) Code equation (14) in symbolic matlab. It will be used later in the assignment.</u>

Theta =

$$\begin{pmatrix}\phi\\\theta\\\psi\end{pmatrix}$$

H(Theta) =

$$\begin{pmatrix}1 & 0 & -\sin(\theta)\\0 & \cos(\phi) & \cos(\theta)\sin(\phi)\\0 & -\sin(\phi) & \cos(\phi)\cos(\theta)\end{pmatrix}$$

$\omega_{\mathrm{bE}} =$

$$\begin{pmatrix}p\\q\\r\end{pmatrix}$$

$\Theta_{\mathrm{dot}} =$

$$\begin{pmatrix} \dfrac{p\cos(\theta) + r\cos(\phi)\sin(\theta) + q\sin(\phi)\sin(\theta)}{\cos(\theta)} \\ q\cos(\phi) - r\sin(\phi) \\ \dfrac{r\cos(\phi) + q\sin(\phi)}{\cos(\theta)} \end{pmatrix}$$

(2) Can you explain why gyroscope measurements are not the same as pitch rate, yaw rate, and roll rate?

Difference between the body and earth frame.

Gyroscope measurements (p, q, r) represent the angular velocity components along the body-fixed axes of the drone, which are directly tied to the drone's own orientation. On the other hand, pitch rate, yaw rate, and roll rate ($\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$) are changes in the Euler angles that describe the drone's orientation in relation to a fixed reference frame, in this case the Earth frame.

(3) These equations (14) lose differentiability when θ is ±π/2 which is called a singularity. Suppose that a drone has tilted up to these angles. The gyroscope measures a large value for p but q, r are both measuring zero. What is the drone's pitch rate, yaw rate, and roll rate?

H:

$$H\left(\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix}\right) = H_\theta$$

When Theta = pi/2:

$$H\left(\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & \cos(\phi) & 0 \\ 0 & -\sin(\phi) & 0 \end{pmatrix}$$

When a singularity occurs, it becomes hard to accurately determine the drone's pitch rate, yaw rate, and roll rate. This is because as can be seen in the matrix above, changes in phi can produce the same affect as pitch making measurnments ambigous. So at singularities, these quantities can not be accurately calculated and singularities should be avoided.

Model Gravity:

Problem.

(1) Rotate the gravity vector from the earth frame to the body frame. Assume that gravity points downwards and has a constant acceleration $9.81 \;\; m/s$.

$$\textbf{(15)} \quad F_g^b = R_E^b F_g^E$$

Write out the gravity vector in body coordinates as a function of ϕ and θ.

Rotational Matrix for phi:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix}$$

Rotational Matrix for theta:

$$\begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

Rotational Matrix for psi:

$$\begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotational Matrix from Earth Frame to Body Frame:

$$\begin{pmatrix} \cos(\psi)\cos(\theta) & \cos(\theta)\sin(\psi) & -\sin(\theta) \\ \cos(\psi)\sin(\phi)\sin(\theta) - \cos(\phi)\sin(\psi) & \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi)\sin(\theta) & \cos(\theta)\sin(\phi) \\ \sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi)\sin(\theta) & \cos(\phi)\sin(\psi)\sin(\theta) - \cos(\psi)\sin(\phi) & \cos(\phi)\cos(\theta) \end{pmatrix} = \text{rotational}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Force of gravity vector in the body frame:

$$\begin{pmatrix} \frac{981\, m \sin(\theta)}{100} \\ -\frac{981\, m \cos(\theta)\sin(\phi)}{100} \\ -\frac{981\, m \cos(\phi)\cos(\theta)}{100} \end{pmatrix} = \begin{pmatrix} \cos(\psi)\cos(\theta) & \cos(\theta)\sin(\psi) & \\ \cos(\psi)\sin(\phi)\sin(\theta) - \cos(\phi)\sin(\psi) & \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi)\sin(\theta) & c \\ \sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi)\sin(\theta) & \cos(\phi)\sin(\psi)\sin(\theta) - \cos(\psi)\sin(\phi) & c \end{pmatrix}$$

(2) Write symbolic matlab that programs calculates (7.1)

The output above shows the results of the symbolic matlab formulation.

(3) An F-15 has a max gross weight of about $82,000\ lbs$. The aircraft is pitched up to $60\ degrees$ pitch angle and rolled by $45\ degrees$. What is the force of gravity F_g^b at this flight condition?

Force of gravity vector in the body frame:

$$\begin{pmatrix} \frac{981\, m \sin(\theta)}{100} \\ -\frac{981\, m \cos(\theta)\sin(\phi)}{100} \\ -\frac{981\, m \cos(\phi)\cos(\theta)}{100} \end{pmatrix}$$

Force of gravity vector in the body frame after plugging in fighter jet mass , 60 degree pitch angle and 45

$$\begin{pmatrix} -111218.926 \\ 295701.977 \\ 182557.419 \end{pmatrix}$$

(4) The same aircraft is at the same condition. It now additionally yaws $90 \ degrees$. What is the force of gravity F_g^b at this flight condition?

The gravity force vector will remain the same as it acts downwards pependicular to the aircraft and as such would ne be affected by the fighter jet yawing.

Translational Kinemetics:

The velocity of the vehicle in an earth frame can be represented as

$$(16) \quad V^E = \begin{bmatrix} V_N \\ V_E \\ V_D \end{bmatrix}$$

This is the ground speed of the UAV. When there is no wind, this is equivalent to the airpseed of the vehicle. The airspeed in body axis is related to the ground speed in the earth frame as follows:

$$(17) \quad V^b = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = R_E^b V^E$$

Problem.

(1) Write symbolic matlab code that takes Θ and V^b and outputs V^E and position in the NED frame p^E.

Theta =

$$\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix}$$

velocity of the vehicle in an earth =

$$\begin{pmatrix} V_N \\ V_E \\ V_D \end{pmatrix}$$

Airspeed of the vehicle in the body axis =

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Rotational Matrix from Earth Frame to Body Frame:

$$\begin{pmatrix} \cos(\psi)\cos(\theta) & \cos(\theta)\sin(\psi) & -\sin(\theta) \\ \cos(\psi)\sin(\phi)\sin(\theta) - \cos(\phi)\sin(\psi) & \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi)\sin(\theta) & \cos(\theta)\sin(\phi) \\ \sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi)\sin(\theta) & \cos(\phi)\sin(\psi)\sin(\theta) - \cos(\psi)\sin(\phi) & \cos(\phi)\cos(\theta) \end{pmatrix}$$

Transposing the matrix to get the Rotational Matrix from Body Frame to Earth Frame:

$$\begin{pmatrix} \cos(\psi)\cos(\theta) & \cos(\psi)\sin(\phi)\sin(\theta) - \cos(\phi)\sin(\psi) & \sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi)\sin(\theta) \\ \cos(\theta)\sin(\psi) & \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi)\sin(\theta) & \cos(\phi)\sin(\psi)\sin(\theta) - \cos(\psi)\sin(\phi) \\ -\sin(\theta) & \cos(\theta)\sin(\phi) & \cos(\phi)\cos(\theta) \end{pmatrix}$$

Multypling the new rotation matrix by the vehicle's speed inthe body frame results in the vehicle's speed ir

$$\begin{pmatrix} V_N \\ V_E \\ V_D \end{pmatrix} = \begin{pmatrix} \cos(\psi)\cos(\theta) & \cos(\psi)\sin(\phi)\sin(\theta) - \cos(\phi)\sin(\psi) & \sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi)\sin(\theta) \\ \cos(\theta)\sin(\psi) & \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi)\sin(\theta) & \cos(\phi)\sin(\psi)\sin(\theta) - \cos(\psi)\sin(\phi) \\ -\sin(\theta) & \cos(\theta)\sin(\phi) & \cos(\phi)\cos(\theta) \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\begin{pmatrix} V_N \\ V_E \\ V_D \end{pmatrix} = \begin{pmatrix} w\ (\sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi)\sin(\theta)) - v\ (\cos(\phi)\sin(\psi) - \cos(\psi)\sin(\phi)\sin(\theta)) + u\cos(\psi)\,\mathrm{cc} \\ v\ (\cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi)\sin(\theta)) - w\ (\cos(\psi)\sin(\phi) - \cos(\phi)\sin(\psi)\sin(\theta)) + u\cos(\theta)\,\mathrm{si} \\ w\cos(\phi)\cos(\theta) - u\sin(\theta) + v\cos(\theta)\sin(\phi) \end{pmatrix}$$

Aerodynamic Forces and Moments

A lumped model tries to combine the effect of many sources of drag and creates a simplified, useful model as follows [3]:

$$F_A^b = -\rho C_T D^4(n_1^2 + n_2^2 + n_3^2 + n_4^2) \begin{bmatrix} K_c & 0 & 0 \\ 0 & K_c & 0 \\ 0 & 0 & K_c \end{bmatrix} V^b \tag{18}$$

Let $K_c - 0.22$. We'll assume no aerodynamics moments.

$$M_A^b = 0 \tag{19}$$

The thrust T_p and the torque Q_p along the propeller axis can be modeled as

$$T_p = \rho \left(\frac{\Omega}{2\pi}\right)^2 D^4 C_T(J)$$

$$Q_p = \rho \left(\frac{\Omega}{2\pi}\right)^2 D^5 C_Q(J),$$

where

$$\rho \triangleq \text{density of air}$$

$$D \triangleq \text{ propeller diameter (m)}$$

$$C_T, C_Q \triangleq \text{ aerodynamic coefficients}$$

$$J \triangleq \frac{V_a}{nD} = \frac{2\pi V_a}{\Omega D}, \text{ advance ratio (unitless)}$$

$$\Omega \triangleq \text{ propeller speed (rad/sec)}$$

Problem.

(1) Write symbolic matlab code that takes V_b and n_1, n_2, n_3, n_4 as an input and outputs F_A^b.

Model for force of the drone based on propeller speed:

$$\begin{pmatrix} -C_T \, \mathrm{D}^4 \, K_c \, \rho \, u \, \sigma_1 \\ -C_T \, \mathrm{D}^4 \, K_c \, \rho \, v \, \sigma_1 \\ -C_T \, \mathrm{D}^4 \, K_c \, \rho \, w \, \sigma_1 \end{pmatrix} = -\rho \, C_T \, \mathrm{D}^4 \, \sigma_1 \begin{pmatrix} K_c & 0 & 0 \\ 0 & K_c & 0 \\ 0 & 0 & K_c \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

where

$$\sigma_1 = {n_1}^2 + {n_2}^2 + {n_3}^2 + {n_4}^2$$

Full Equations of Motion:

Write a symoblic matlab scripts that combines all equations and implements the following equations:

(20) $\dot{p}^E = V^E$

(21) $\dot{\Theta} = H(\Theta)^{-1} \omega_{b/E}$

(22) $\dot{V}^b = \frac{1}{m}\left(\vec{F} + \vec{F}^b_g + \vec{F}^b_A - \omega_{b/E} \times m V^b\right)$

$$\text{(23)} \quad \dot{\omega}_{b/E} = J^{-1}\left(\vec{M} - \omega_{b/E} \times J\omega_{b/E}\right)$$

The open loop aircraft dynamics take $\vec{F} = [X; Y; Z]$ and $\vec{M} = [L; M; N]$ as force and moment input from the motors and outputs p^E, V^b, Θ, $\omega_{b/E}$.

The 12 states of the drone:

$$\mathrm{x} = \begin{pmatrix} p_n \\ p_e \\ p_d \\ \phi \\ \theta \\ \psi \\ u \\ v \\ w \\ p \\ q \\ r \end{pmatrix}$$

The 12 state derivatives of the drone:

$$x_{\mathrm{dot}} = \begin{pmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix}$$

The kinematic equations for tranlational velocity:

$$\begin{pmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \end{pmatrix} = \begin{pmatrix} \cos(\psi)\cos(\theta) & \cos(\psi)\sin(\phi)\sin(\theta) - \cos(\phi)\sin(\psi) & \sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi)\sin(\theta) \\ \cos(\theta)\sin(\psi) & \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi)\sin(\theta) & \cos(\phi)\sin(\psi)\sin(\theta) - \cos(\psi)\sin(\phi) \\ -\sin(\theta) & \cos(\theta)\sin(\phi) & \cos(\phi)\cos(\theta) \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

The kinematic equations for rotational velocity:

$$\begin{pmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{pmatrix} = \begin{pmatrix}1 & \frac{\sin(\phi)\sin(\theta)}{\cos(\theta)} & \frac{\cos(\phi)\sin(\theta)}{\cos(\theta)}\\ 0 & \cos(\phi) & -\sin(\phi)\\ 0 & \frac{\sin(\phi)}{\cos(\theta)} & \frac{\cos(\phi)}{\cos(\theta)}\end{pmatrix}\begin{pmatrix}p\\ q\\ r\end{pmatrix}$$

The Translational Dynamics:

$$\begin{pmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{pmatrix} = \begin{pmatrix} r\,v - q\,w + \frac{f_x}{m}\\ p\,w - r\,u + \frac{f_y}{m}\\ q\,u - p\,v + \frac{f_z}{m}\end{pmatrix}$$

The Rotational Dynamics:

$$\begin{pmatrix}\dot{p}\\ \dot{q}\\ \dot{r}\end{pmatrix} = \begin{pmatrix} L - \frac{J_{\mathrm{xz}}\,\sigma_3}{\sigma_1} - \frac{J_z\,\sigma_2}{\sigma_1}\\ M - \frac{p\,\left(J_{\mathrm{xz}}\,p - J_z\,r\right) + r\,\left(J_x\,p - J_{\mathrm{xz}}\,r\right)}{J_y}\\ N - \frac{J_x\,\sigma_3}{\sigma_1} - \frac{J_{\mathrm{xz}}\,\sigma_2}{\sigma_1}\end{pmatrix}$$

where

$$\sigma_1 = {J_{\mathrm{xz}}}^2 - J_x\,J_z$$

$$\sigma_2 = q\,\left(J_{\mathrm{xz}}\,p - J_z\,r\right) + J_y\,q\,r$$

$$\sigma_3 = q\,\left(J_x\,p - J_{\mathrm{xz}}\,r\right) - J_y\,p\,q$$

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X_dot =
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$$
\begin{pmatrix}
w\ (\sin(\phi)\sin(\psi)+\cos(\phi)\cos(\psi)\sin(\theta)) - v\ (\cos(\phi)\sin(\psi)-\cos(\psi)\sin(\phi)\sin(\theta)) + u\cos(\psi)\cos(\theta) \\
v\ (\cos(\phi)\cos(\psi)+\sin(\phi)\sin(\psi)\sin(\theta)) - w\ (\cos(\psi)\sin(\phi)-\cos(\phi)\sin(\psi)\sin(\theta)) + u\cos(\theta)\sin(\psi) \\
w\cos(\phi)\cos(\theta) - u\sin(\theta) + v\cos(\theta)\sin(\phi) \\
r\,v - q\,w + \frac{f_x}{m} \\
p\,w - r\,u + \frac{f_y}{m} \\
q\,u - p\,v + \frac{f_z}{m} \\
p + \frac{r\cos(\phi)\sin(\theta)}{\cos(\theta)} + \frac{q\sin(\phi)\sin(\theta)}{\cos(\theta)} \\
q\cos(\phi) - r\sin(\phi) \\
\frac{r\cos(\phi)}{\cos(\theta)} + \frac{q\sin(\phi)}{\cos(\theta)} \\
L - \frac{J_{\mathrm{xz}}\,\sigma_3}{\sigma_1} - \frac{J_z\,\sigma_2}{\sigma_1} \\
M - \frac{p\,\sigma_4 + r\,\sigma_5}{J_y} \\
N - \frac{J_x\,\sigma_3}{\sigma_1} - \frac{J_{\mathrm{xz}}\,\sigma_2}{\sigma_1}
\end{pmatrix}
$$

where

$$\sigma_1 = {J_{\mathrm{xz}}}^2 - J_x\,J_z$$

$$\sigma_2 = q\,\sigma_4 + J_y\,q\,r$$

$$\sigma_3 = q\,\sigma_5 - J_y\,p\,q$$

$$\sigma_4 = J_{\mathrm{xz}}\,p - J_z\,r$$

$$\sigma_5 = J_x\,p - J_{\mathrm{xz}}\,r$$

Linearization:

Use symbolic matlab to linearize these dynamics around a hover equilibrium. Present the linear state-space model and clearly specify what is the state vector and the input vector.

After Taking the Jacobian:

$$
\left(\begin{array}{llllll}
0 & 0 & 0 & v\,\sigma_1 + w\,(\sigma_{11} - \sigma_6) & \cos(\psi)\,\sigma_3 & w\,(\sigma_{10} - \sigma_7) - v\,\sigma \\
0 & 0 & 0 & -v\,(\sigma_{10} - \sigma_7) - w\,\sigma_2 & \sin(\psi)\,\sigma_3 & w\,\sigma_1 - v\,(\sigma_{11} - \sigma_6 \\
0 & 0 & 0 & \cos(\theta)\,(v\cos(\phi) - w\sin(\phi)) & -u\cos(\theta) - w\cos(\phi)\sin(\theta) - v\sin(\phi)\sin(\theta) & \\
0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & \frac{\sin(\theta)\,\sigma_4}{\cos(\theta)} & \frac{\sigma_5}{\cos(\theta)^2} & \\
0 & 0 & 0 & -r\cos(\phi) - q\sin(\phi) & 0 & \\
0 & 0 & 0 & \frac{\sigma_4}{\cos(\theta)} & \frac{\sin(\theta)\,\sigma_5}{\cos(\theta)^2} & \\
0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 &
\end{array}\right.
$$

where

$$\sigma_1 = \sin(\phi)\sin(\psi) + \cos(\phi)\cos(\psi)\sin(\theta)$$

$$\sigma_2 = \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi)\sin(\theta)$$

$$\sigma_3 = w\cos(\phi)\cos(\theta) - u\sin(\theta) + v\cos(\theta)\sin(\phi)$$

$$\sigma_4 = q\cos(\phi) - r\sin(\phi)$$

$$\sigma_5 = r\cos(\phi) + q\sin(\phi)$$

$$\sigma_6 = \cos(\psi)\sin(\phi)\sin(\theta)$$

$$\sigma_7 = \cos(\phi)\sin(\psi)\sin(\theta)$$

$$\sigma_8 = J_{xz}\,p + J_y\,r - J_z\,r$$

$$\sigma_9 = J_y\,p - J_x\,p + J_{xz}\,r$$

$$\sigma_{10} = \cos(\psi)\sin(\phi)$$

$$\sigma_{11} = \cos(\phi)\sin(\psi)$$

$$\sigma_{12} = \frac{J_{xz}\,q\,(J_x - J_y + J_z)}{\sigma_{13}}$$

After Plugging in x = 0 into the jacobian:

$$\begin{pmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_n \\ p_e \\ p_d \\ \phi \\ \theta \\ \psi \\ u \\ v \\ w \\ p \\ q \\ r \end{pmatrix}$$